Improvement in the plastic rotation evaluation by means of fracture mechanics concepts

M. Corrado, A. Carpinteri, M. Paggi & G. Mancini
Politecnico di Torino, Department of Structural Engineering and Geotechnics, Torino, Italy

ABSTRACT: The well-known Cohesive Crack Model describes strain localization with a softening stress variation in concrete members subjected to tension. Based on the assumption that strain localization also occurs in compression, the Overlapping Crack Model, analogous to the cohesive one, is proposed to simulate material compenetration due to crushing. By applying this model, it is possible to describe the size effects in compression in a rational way. The two aforementioned elementary models are then merged into a more complex algorithm based on the finite element method, able to describe both cracking and crushing growths during loading processes in RC members. With this algorithm in hand, it is possible to investigate on the influence of the reinforcement percentage and/or the structural size of RC beams, with special attention to their rotational capacity. The obtained results evidence that the prescriptions concerning the plastic rotations provided by codes of practice, not taking into account the scale effects, are not conservative in the case of large structural sizes.

1 INTRODUCTION

The development of considerable ductility in the ultimate limit state is a key parameter for the design of reinforced concrete (RC) beams in bending. The interest in ductility was formerly connected with the diffusion of plastic analysis in the design of reinforced concrete structures (Macchi 1969). In this context, in fact, the rotational capacity is required to allow the bending moment redistribution in statically indeterminate structures. The ductility contributes to satisfy many other requirements, absolutely necessary in order to guarantee the structural safety, as, e.g., to provide robustness, to give warning of incipient collapse by the development of large deformation prior to collapse and to enable major distortions and energy dissipation during earthquakes.

Due to its complexity, this phenomenon was firstly analyzed from an experimental point of view. As a result of the “Indeterminate Structures Commission” of the Comité Européen du Béton (1961), published in Baker & Amakarone (1967), an empirical hyperbolic relationship between plastic rotation and relative neutral axis depth, \( x/d \), was proposed to solve the problem of plastic rotation evaluation for practical purposes. A second fundamental contribution comes from the experimental and analytical research carried out in the early 1980s at the University of Stuttgart, by the group coordinated by Professor Eligehausen (Langer 1987, Eligehausen & Langer 1987). With respect to the foregoing empirical relationship, two aspects were introduced: firstly, an increasing branch for low values of \( x/d \), due to steel rupture, and, secondly, the explicit consideration of two classes of ductility for the reinforcement.

Further improvement in the analysis of the behaviour of RC members in bending was marked by the pioneering paper by Hillerborg (1990) who introduced the concept of strain localization in concrete in compression. According to this approach, when the ultimate compressive strength is achieved, a strain localization takes place within a characteristic length proportional to the depth of the compressed zone. This model permits to address the issue of size effects, although the definition of the length over which the strain localization occurs is a free parameter and its value is not defined on the basis of theoretical arguments. Afterwards, several models have been developed on the basis of the Stuttgart and the Hillerborg proposals, emphasizing some more specific aspects (Cosenza et al. 1991, Tue et al. 1996, Bigaj & Walraven 2002, Fantilli et al. 2007).

In order to assess the rotational capacity of RC beams, the Eurocode 2 (2003) provides a design diagram relating the admissible plastic rotation to the relative neutral axis position. From the analysis of these curves, it appears that the size-scale effects on the rotational capacity of RC beams are not considered,
although the dependence on the structural dimension was recognized in several experimental tests (Mattock 1965, Corley 1966, Siviero 1974, Bosco & Debernardi 1992, Bigaj & Walraven 1993).

In the case of concrete specimens subjected to uniaxial compression, several experiments (Van Vliet & Van Mier 1996, Jansen & Shah 1997) reveal that the size-scale effects are due to two interconnected phenomena: the strain localization after the peak load and the consequent energy dissipation over a surface, the value of which referred to a unitary surface can be considered as a material parameter. Based on these evidences, Carpinteri et al. (2007) have recently proposed to model the process of concrete crushing using an approach analogous to the Cohesive Crack Model, which is adopted for the tensile behaviour of concrete. The former approach is referred to as the Overlapping Crack Model (Carpinteri et al. 2007), which assumes a stress-displacement law for the post-peak behaviour. In tension, the localized displacement is represented by a tensile crack opening, while in compression it is represented by a compenetration, as clearly shown in Figure 1.

In this paper, we propose a new numerical method able to describe the nonlinear behaviour of RC members during both fracturing and crushing. Firstly, the Cohesive Crack Model for concrete in tension and the Overlapping Crack Model for concrete in compression are introduced. Then, a numerical algorithm based on the finite element method is presented for the analysis of intermediate situations ranging from pure concrete members to over-reinforced beams. It is assumed that the fracturing and crushing processes are fully localized along the mid-span cross-section of the representative structural element in bending. This assumption, fully consistent with the physics of the crushing phenomenon, also implies that only one equivalent main tensile crack is considered. In order to validate the proposed model, a comparison between the numerical predictions and the experimental results for the beams tested by Bosco and Debernardi (1992) is carried out. Finally, as a result of a parametric investigation on the influence of the structural dimension and of the steel percentage, a practical diagram showing the size-scale effects on ductility is reported, in terms of plastic rotation as a function of the relative neutral axis position.

2 NUMERICAL INVESTIGATION

In this section, a new model based on Fracture Mechanics concepts is proposed for the evaluation of the rotational capacity of RC beams in bending. Let us consider a portion of a RC beam subjected to a bending moment \( M \). This element, having a span to depth ratio equal to one, is representative of the central zone of the beam where a plastic hinge formation takes place. Note, incidentally, that the analysis of this structural element is also consistent with the prescriptions reported in the Eurocode 2 (2003). We also assume that the mid-span cross-section of this element is fully representative of its mechanical behaviour. The stress distribution in this cross-section is linear-elastic until the tensile stress at the intrados reaches the concrete tensile strength. When this threshold is reached, a cohesive crack propagates from the beam intrados towards its extrados. Correspondingly, the applied moment increases. Outside the crack, the material is assumed to behave linear-elastically. According to the well-known Cohesive Crack Model (Carpinteri 1985), the stresses in the cohesive zone are assumed to be decreasing functions of the crack opening displacement until a critical value of crack opening is reached.

On the other hand, concrete crushing takes place when the maximum stress in compression reaches the concrete compressive strength. After that, damage will be described by a compenetration of the two half-beams, representing in this way the localization of the dissipated energy (Fig. 2b). Larger is the compenetration, also referred to as overlapping in the sequel, lower are the transferred forces along the damaged zone.

2.1 Cohesive Crack Model for the description of concrete fracturing

A pioneering model for the analysis of nonlinear crack propagation in concrete was proposed by Hillerborg et al. (1976) with the name of Fictitious Crack Model. Subsequently, an updated algorithm was implemented by Carpinteri (1985, 1989), with the terminology of Cohesive Crack Model, in order to study the ductile to brittle transition in plane concrete beams in bending. According to this model, the constitutive law used for
the non-damaged zone is a $\sigma - \varepsilon$ linear-elastic relationship up to the tensile strength, $\sigma_{t,u}$. In the process zone, the damaged material is still able to transfer a tensile stress across the crack surfaces. The cohesive stresses are considered to be decreasing functions of the crack opening, $w^t$, as follows:

$$\sigma_t = \sigma_{t,u} \left(1 - \frac{w^t}{w_c^t}\right),$$

where $w^t$ is the crack opening, $w_c^t$ is the critical value of the crack opening corresponding to the condition $\sigma_t = 0$, and $\sigma_{t,u}$ is the ultimate tensile strength of concrete.

The area defined by the stress vs. displacement curve represents the fracture energy $G_F$.

2.2 Overlapping Crack Model for the description of concrete crushing

The most frequently adopted constitutive laws for concrete in compression describe the material behaviour in terms of stress and strain (elastic-perfectly plastic, parabolic-perfectly plastic, Sargin’s parabola, etc). Such approaches imply that the energy is dissipated in a volume, whereas experimental results reveal that the energy is substantially dissipated over a surface as a result of strain localization, (Van Vliet & Van Mier 1996, Jansen & Shah 1997). Hillerborg (1990) firstly proposed to model the crushing phenomenon as a strain localization over a length proportional to the depth of the compressed zone. This condition, however, does not permit to formulate a material constitutive law fully describing the mechanical response of concrete in compression.

In the present formulation, we adopt the stress-displacement relationship proposed by Carpinteri et al. (2007) and then considered also in Corrado (2007) between the compressive stress and the compenetration, in close analogy with the cohesive model. The main hypotheses are:

1. The constitutive law used for the undamaged material is a linear-elastic stress-strain relationship, see Figure 3a.
2. The crushing zone develops when the maximum compressive stress reaches the concrete compressive strength.
3. The process zone is perpendicular to the main compressive stress.
4. The damaged material in the process zone is assumed to be able to transfer compressive stresses between the overlapping surfaces. They are assumed to be decreasing functions of the compenetration, $w_c$ (see Fig. 3b):

$$\sigma_c = \sigma_{c,u} \left(1 - \frac{w_c}{w_c^c}\right),$$

where $w_c$ is the compenetration, $w_c^c$ is the critical value of the compenetration corresponding to the condition of $\sigma_c = 0$, and $\sigma_{c,u}$ is the ultimate compressive strength.

It is worth noting that the crushing energy, $G_F^c$, defined as the area below the post-peak softening curve, is now a true material parameter, since it is not affected by the structural dimension.

An empirical formulation for calculating the crushing energy has recently been proposed by Suzuki et al. (2006), based on the results of compression tests carried out on plane and transversal reinforced concrete specimens. In this study, the crushing energy is computed according to the following empirical equation, which considers the confined concrete compressive strength by means of the stirrups yield strength and the stirrups volumetric content:

$$G_F^c = \frac{G_F}{\sigma_c} + 10000 \frac{k_s^2 p_c}{\sigma_c^2},$$

where $\sigma_{c,0}$ is the average concrete compressive strength, $k_s$ is a parameter depending on the stirrups strength and on the volumetric percentage, and $p_c$ is the effective lateral pressure.

The crushing energy for unconfined concrete, $G_F^{c,0}$, can be calculated using the following expression:

$$G_F^{c,0} = 80 - 50 k_b,$$

where the parameter $k_b$ depends on the concrete compressive strength (see Suzuki et al. 2006).

It is worth noting that $G_F^{c}$ is between 2 and 3 orders of magnitude higher than $G_F$. Finally, we remark that the critical values for crushing compenetration and crack opening are approximately equal to $w_c^c \approx 1$ mm (see also the experimental results by Jansen and Shah 1997) and $w_t^c \approx 0.1$ mm, respectively.

2.3 Numerical algorithm

A discrete form of the elastic equations governing the mechanical response of the two half-beams is herein introduced in order to develop a suitable algorithm for the analysis of intermediate situations where
both fracturing and crushing phenomena take place. According to the finite element method, the mid-span cross-section of the beam is subdivided into finite elements with \( n \) nodes (Fig. 4). In this scheme, cohesive and overlapping stresses are replaced by equivalent nodal forces by integrating the corresponding tractions over each finite element size. Such nodal forces depend on the nodal opening or closing displacements according to the cohesive or overlapping softening laws previously introduced.

With reference to Figure 4, the horizontal forces \( F \) acting along the mid-span cross-section are given by:

\[
\{F\} = [K_w]\{w\} + [K_m]M
\]

where \( \{F\} \) is the vector of nodal forces, \([K_w]\) is the matrix of the coefficients of influence for the nodal displacements, \( \{w\} \) is the vector of nodal displacements, \([K_m]\) is the vector containing the coefficients of influence for the applied moment, and \( M \) is the applied moment.

The coefficients of influence, \( K_{wi} \), present the physical dimension of a stiffness and are computed a priori with a finite element analysis by imposing a unit displacement to each of the nodes shown in Figure 4.

In the generic situation shown in Figure 5, the following equations can be considered:

\[
F_i = 0; \quad \text{for } i = 1, 2, \ldots, (j-1); \ j \neq r \quad (5a)
\]

\[
F_i = F_{i,w} \left( 1 - \frac{w_i^t}{w_c^t} \right); \quad \text{for } i = j, \ldots, (m-1); \ i \neq r \quad (5b)
\]

\[
w_i^t = 0; \quad \text{for } i = m, \ldots, p \quad (5c)
\]

\[
F_i = F_{i,c} \left( 1 - \frac{w_i^c}{w_c^c} \right); \quad \text{for } i = (p+1), \ldots, n \quad (5d)
\]

\[
F_r = f(w); \quad \text{for } i = r \quad (5e)
\]

Equation (5e) represents the relationship between the closing force exerted by the reinforcing steel and the crack opening at the reinforcement level. Such a law is determined on the basis of the bond-slip behaviour of concrete and steel.

Equations (4) and (5) constitute a linear algebraic system of \((2n)\) equations and \((2n+1)\) unknowns, namely \( \{F\}, \{w\} \) and \( M \). A possible additional equation can be introduced: we can set either the force in the fictitious crack tip, \( m \), equal to the ultimate tensile force, or the force in the fictitious crushing tip, \( p \), equal to the ultimate compressive force. In the numerical scheme, we choose the situation which is closer to one of the two possible critical conditions. This criterion will ensure the uniqueness of the solution on the basis of physical arguments. The driving parameter of the process is the tip that in the considered step has reached the limit resistance. Only this tip is moved when passing to the next step.

3 EXPERIMENTAL COMPARISON AND NEW PROPOSAL FOR DESIGN CODES

In this section, a comparison between the numerical predictions using the cohesive/overlapping model and the experimental results of the tests carried out by Bosco and Debernardi (1992) is proposed. In order to obtain a consistent comparison, the numerical simulations have been carried out by modelling the beam portion positioned at the mid-span of the beam. Such an element is characterized by a span to depth ratio equal to one. The rotations of such portion were experimentally determined as functions of the applied bending moment.

Numerical and experimental moment-rotation curves are then compared in the Figures 6a,b,c for different beam depths and different steel percentages. These diagrams put into evidence that the maximum rotation is a decreasing function of the tensile reinforcement ratio and of the beam depth. In the case
of low steel percentages, the mechanical behaviour is characterized by the reinforcement yielding and the mechanical response is almost plastic. By increasing the amount of reinforcement, the contribution of concrete crushing becomes more and more evident with the appearance of a softening branch at the end of the plastic plateau. This is an important feature of the proposed model, which also permits to follow unstable softening branches with positive slope, (snap-back). This is possible by controlling the loading process through the length of the tensile crack and the extension of the fictitious crushing zone, rather than by the external load. A good agreement is obtained between numerical and experimental results for all the tested beams.

4 CONCLUSIONS

In the present paper, a numerical algorithm has been proposed for the analysis of the behaviour of RC elements in bending. To this aim, all the principal nonlinear contributions are taken into account for an accurate evaluation of the moment-rotation diagram. With this tool in hand, the simulation of the mechanical behaviour of all the intermediate situations ranging from pure concrete to over-reinforced concrete beams can be made. The following main conclusions can be drawn from the comparison between numerical predictions and experimental results:

1. The introduced constitutive law for concrete in compression through the Overlapping Crack Model

With reference to the moment-rotation diagrams, the plastic component of the total rotation can be obtained as the difference between the ultimate rotation and the rotation corresponding to the reinforcement yielding. According to the definition proposed by Hillerborg (1990) and Pecce (1997), the ultimate rotation is the rotation beyond which the moment starts descending rapidly.

The results of the parametric analysis can be summarized in a plastic rotation vs. relative neutral axis position, $x/d$, diagram (see Fig. 7). This is also consistent with the practical prescriptions of the Eurocode 2. As can be readily seen, the rotational capacity is a nonlinear function of the relative neutral axis position. In a first stage, for $x/d$ up to approximately 0.05, the plastic rotation is an increasing function of $x/d$, since beam failure is mainly governed by steel yielding. Then, when crushing comes into play by increasing the amount of reinforcement, the plastic rotation turns out to be a decreasing function of $x/d$. Moreover, note that size-scale effects are particularly relevant, since the higher is the beam depth, the lower is the rotational capacity.
permits to describe the nonlinear behaviour of concrete considering the effect of the structural dimension and to describe the descending branch of the moment-rotation diagram, as shown in Figure 6. For large structural sizes and/or high reinforcement percentages, this softening branch can have a positive slope, leading to a snap-back instability.

2. Referring to Figure 6, it is possible to state that the proposed algorithm catches the experimental results (Bosco & Debernardi 1992) by varying both the structural dimension and the steel percentage.

3. Independently of the reinforcement ratio, the behaviour becomes more and more brittle by increasing the beam depth, with a progressively reduction of the ultimate rotation.

4. According to the Eurocode 2 (2003), the plastic rotation of RC beams is considered to be a function of the neutral axis position only. In order to improve the code provisions, the effect of the structural dimension should be explicitly taken into account by considering different design curves, as, for instance, those proposed in Figure 7.

REFERENCES


